Étale & smooth morphisms.

The cotangent complex can also be used to understand étale and smooth morphism.

Prop: Let $f: X \to Y$ be a finitely presented morphism between objects of Sch, we have

(i) f is étale, i.e. for some Zariski cover $\coprod_I S_i \to X$ each composite $S_i \to Y$ is étale, iff $T^*(X/Y) \simeq 0$.

(ii) f is smooth (defined similarly for schemes as above), iff $T^*(X/Y) \in \mathrm{Vect}(X)$ i.e. $T^*(X/Y)$ is dualizable in $QCoh(X)^{\leq 0}$, and f is f.p.

Pf: We claim the result can be reduced to the statement for affine schemes. Indeed since $f: X \to Y$ is étale if there exists a cover $\coprod_I S_i \to X$ s.t. each $f_i: S_i \to Y$ is étale we can reduce to the case where X is affine. Now f_i is affine representable, since $Y \to Y \times Y$ is affine, so by considering a cover $\coprod_J T_j \to Y$ we can consider:

$$\begin{array}{ccc} S_i \times_Y T_j & \to & T_j \\ g \downarrow & & \downarrow \\ S_i & \to & Y \end{array}$$

and since $g^* T^*(S_i/Y) = T^*(S_i \times_Y T_j / T_j)$. It is enough to consider Y affine. (Since being f.p. is stable under base change.)

Assume $f: S = \mathrm{Spec}\, B \to \mathrm{Spec}\, A = T$ is étale, i.e. $A \to B$ is flat & $H^0(A) \to H^0(B)$ is étale.

Then

$$\begin{array}{ccc} A & \to & B \\ \downarrow & \ulcorner & \downarrow \\ H^0(A) & \to & H^0(B) \end{array}$$

is a push-out diagram. So we have: $\mathbb{L}_{B/A} \otimes_B H^0(B) \simeq \mathbb{L}_{H^0(B)/H^0(A)}$

Since for usual comm. algs. we have. $\mathbb{L}_{H^0(B)/H^0(A)}$ recovers $L_{H^0(B)/H^0(A)}$ as defined in [Stacks-project] for instance. Then $H^0(A) \to H^0(B)$ étale implies. $L_{H^0(B)/H^0(A)} = 0$. (Tag 08RZ).

Now one has a spectral sequence: $H^p(H^0(B) \otimes_{H^0(B)} H^q(\mathbb{L}_{B/A})) \Rightarrow H^{p+q}(H^0(B) \otimes_B \mathbb{L}_{B/A})$

We notice that for $H^{-1}(\mathbb{L}_{B/A}) \neq 0$ one would need $\mathrm{Tor}^i(H^0(B), H^{-i-1}(\mathbb{L}_{B/A})) \neq 0$ for all $i \geq 1$. In particular, $H^{-k}(\mathbb{L}_{B/A}) \neq 0$ for $k << 0$. [HA, 7.4.3.18].

However, $A \to B$ f.p. $\Rightarrow$ $\mathbb{L}_{B/A}$ is perfect. So $H^{-1}(\mathbb{L}_{B/A}) = 0$.

By induction one has $H^i(\mathbb{L}_{B/A}) = 0$ $\forall i \in \mathbb{Z}$. $\blacktriangle$

Let's now assume $\mathbb{L}_{B/A} \simeq 0$. We will use induction on $n \geq 0$ to prove that $\tau^{\geq -n}(A) \to \tau^{\geq -n}(B)$ is étale. for all $n \geq 0$.

~~For $H^0(A) \to H^0(B)$, since $H^0(\mathbb{L}_{H^0(B)/H^0(A)}) = H^0(\mathbb{L}_{B/A})$.~~

~~the result follows from this implies that for any~~

Consider the fiber sequences:

$$\begin{array}{ccccc} \mathbb{L}_{H^0(A)/A} \underset{H^0(A)}{\otimes} H^0(B) & \to & \mathbb{L}_{H^0(B)/A} & \to & \mathbb{L}_{H^0(B)/H^0(A)} \\ & & \uparrow & & \\ \mathbb{L}_{B/A} \underset{B}{\otimes} H^0(B) & \to & \mathbb{L}_{H^0(B)/A} & \to & \mathbb{L}_{H^0(B)/B}. \end{array}$$

Since $B \xrightarrow{\alpha} H^0(B)$ has $\mathrm{Cofib}(\alpha) \in \mathrm{Mod}_B^{\leq -2}$ this implies:

$$H^i(\mathbb{L}_{H^0(B)/B}) \simeq H^i(H^0(B) \underset{B}{\otimes} \mathrm{Cofib}(f)) \quad \text{for } i \geq -3.$$

So $H^i(\mathbb{L}_{H^0(B)/B}) = 0$ for $i \geq -1$.

Since $\mathbb{L}_{B/A} \simeq 0 \Rightarrow \tau^{\geq -1}(\mathbb{L}_{H^0(B)/H^0(A)}) = 0$.

Now, $\tau^{\geq -1}(\mathbb{L}_{H^0(B)/H^0(A)}) = NL_{H^0(B)/H^0(A)}$ the naive cotangent complex, by inspecting its universal property. So $H^0(A) \to H^0(B)$ is étale by [Stacks, Def'n 10.143.1].

moreover:

One can check that $\tau^{\leq -n}(\mathbb{L}_{\tau^{\leq -n}(B)/\tau^{\leq -n}(A)}) \simeq \tau^{\leq -n}(\mathbb{L}_{B/A}) \simeq 0$.

So given $M \in \mathrm{Mod}^{\heartsuit}_{H^0(B)}$ and a map $\gamma : \mathbb{L}_{\tau^{\leq -n}(B)/\tau^{\leq -n}(A)} \to M[n+1]$

We can consider the square-zero extension $\tilde{f}$ depicted as:

$$\begin{array}{ccc} \tau^{\leq -n}(A) & \longrightarrow & \tau^{\leq -n}(B) \oplus M[n+1] \\ \downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow \\ \tau^{\leq -n}(B) & \longrightarrow & \tau^{\leq -n}(B) \end{array}$$

However, $\tilde{f}$ is determined by $\tilde{f} : B \to \tau^{\leq -n}(B) \oplus M[n]$. which by the Thm from last time is determined by $\mathbb{L}_{B/A} \to M[n+1]$. which is 0 since $\mathbb{L}_{B/A} = 0$.

Thus, one has that $\mathbb{L}_{\tau^{\leq -n}(B)/\tau^{\leq -n}(A)} \in \mathrm{Mod}^{\leq -n-1}_{\tau^{\leq -n}(B)}$, i.e.

$$\tau^{\geq -n-1}(\mathbb{L}_{\tau^{\leq -n}(B)/\tau^{\leq -n}(A)}) \simeq 0.$$

Now we consider the sequences:

$$\begin{array}{ccccc} \mathbb{L}_{\tau^{\leq -n}(A)} \otimes_{\tau^{\leq -n}(A)} \tau^{\leq -n+1}(A) & \to & \mathbb{L}_{\tau^{\leq -n+1}(A)} & \to & \mathbb{L}_{\tau^{\leq -n+1}(A)/\tau^{\leq -n}(A)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{\tau^{\leq -n}(B)} \otimes_{\tau^{\leq -n}(B)} \tau^{\leq -n+1}(B) & \to & \mathbb{L}_{\tau^{\leq -n+1}(B)} & \to & \mathbb{L}_{\tau^{\leq -n+1}(B)/\tau^{\leq -n}(B)} \end{array}$$

by tensoring the row above by $(-) \otimes_{\tau^{\leq -n+1}(A)} \tau^{\leq -n+1}(B)$, which is étale.

one notices that the left & middle columns become $(n+1)$-connected.

So $H^{-n-1}(\mathbb{L}_{\tau^{\leq -n+1}(A)/\tau^{\leq -n}(A)} \otimes_{\tau^{\leq -n+1}(A)} \tau^{\leq -n+1}(B)) \underset{(\Delta)}{\simeq} H^{-n-1}(\mathbb{L}_{\tau^{\leq -n+1}(B)/\tau^{\leq -n}(B)})$.

Exercise: $H^{-n-1}(\mathbb{L}_{\tau^{\leq -n+1}(A)/\tau^{\leq -n}(A)}) \simeq H^{-n}(A) \underset{(\Delta)}{\Rightarrow} H^{-n}(A) \otimes_{H^0(A)} H^0(B) = H^{-n}(B)$.

Go back to discussion at $QCoh(\mathcal{X})$ for a prestack & $T^*\mathcal{X}$ as an object of $QCoh(\mathcal{X})$. (pg. 5 Talk. 18).

The following result follows from tracing the definitions & using the connectivity estimates.

<u>Lemma:</u> Let $\mathcal{X} \in Stk_{\leq 1}$, ie. a 1-truncated stack. (maybe we need to assume algebraic?) and consider:

$$\iota : {}^{der}({}^{cl}\mathcal{X}) \to \mathcal{X}, \quad \text{where.} \quad {}^{der}({}^{cl}\mathcal{X}) := {}^{L}LKE_{{}^{cl}Sch^{aff} \hookrightarrow Sch^{aff}}({}^{cl}\mathcal{X}).$$

Then one has comparison maps:

$$\iota^* T^*\mathcal{X} \xrightarrow{\phi^1_{\mathcal{X}}} T^*\,{}^{der}({}^{cl}\mathcal{X}) \xrightarrow{\phi^2_{\mathcal{X}}} {}^{cl}T^*\,{}^{cl}\mathcal{X}, \quad \text{where.}$$

$L^{cl}\mathcal{X} = {}^{cl}T^{*cl}\mathcal{X}$ denotes the cotangent complex of an algebraic stack in the classical sense. (see [Stacks,]).

Then: $\phi^1_{\mathcal{X}}$ is 1-connective.

$\phi^2_{\mathcal{X}}$ is 2-connective.

In particular, for any 1-truncated classical algebraic stack $\mathcal{X}_0$ one has:

$$\tau^{\geq -1}(T^*\,{}^{der}(\mathcal{X}_0)) \simeq \tau^{\geq -1}({}^{cl}T^*\mathcal{X}_0).$$

<u>Cor:</u> For any smooth classical scheme Z one has;

$$T^*\,{}^{der}Z \simeq {}^{cl}T^*Z[0], \quad \text{where } {}^{cl}T^*Z \text{ denotes its cotangent sheaf. (v. 6.1).}$$

<u>Deformation theory.</u>

The notion of square-zero extension allows one to define a further condition on prestacks.

Consider $x : S \to \mathcal{X}$ a point in a prestack $\mathcal{X}$ and

$$T^*(S) \xrightarrow{\gamma} \mathcal{F} \in QCoh(S)^{\leq -1}_{T^*(S)/}.$$

This gives a sq-zero extension of S:

$$S \hookrightarrow S' := S \coprod_{S_{\mathcal{F}}} S \quad \text{described last time.}$$
(pr. and d)

<u>Def'n:</u> $\mathcal{X}$ is said to be <u>infinitesimally cohesive</u> if for all (x, S, γ) as above the canonical map:

$$Maps_{S/}(S', \mathcal{X}) \xrightarrow{\simeq} Maps(S, \mathcal{X}) \underset{Maps(S_{\mathcal{F}}, \mathcal{X})}{\overset{\text{is } x}{\times}} Maps(S, \mathcal{X})$$

is an isomorphism.

There is a characterization of prestacks which admit a cotangent complex and are infinitesimally cohesive.

Recall that a map $S \to T$ of affine schemes is a <u>nilpotent embedding</u> if

$$^{cl}S \hookrightarrow {}^{cl}T \text{ is closed } \& \ \mathcal{I}_{^{cl}S \hookrightarrow {}^{cl}T}^{n} = 0 \text{ for some } n \geq 1.$$
(ideal of def'n.)

<u>Prop:</u> Let $\mathcal{X}$ be a convergent prestack, then.

$\mathcal{X}$ admits a cotangent complex & is infinitesimally cohesive. $\Leftrightarrow$ for every $S_1' \coprod_{S_1} S_2$ in Sch^{aff} where $S_1 \to S_1'$ is a nilpotent embedding the map (☆) is an isom.

$$(☆) \quad Maps(S_1' \coprod_{S_1} S_2, \mathcal{X}) \to Maps(S_1', \mathcal{X}) \underset{Maps(S_1, \mathcal{X})}{\times} Maps(S_2, \mathcal{X}).$$

The reason why one has this result is essentially the following fact which says that we can understand any nilpotent embedding as a series of square-zero extensions.

Thm: Let $S \to T$ be a nilpotent embedding of (affine) schemes, then there exists a sequence:

$$S = S_0^0 \hookrightarrow S_0^1 \hookrightarrow \cdots \hookrightarrow S_0^n =: S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots \hookrightarrow T,$$

s.t. (i) each $S_0^i \hookrightarrow S_0^{i+1}$ and $S_j \hookrightarrow S_{j+1}$ has a structure of a square-zero extension;

(ii) for every $j \geq 0$, the map $S_j \to T$ induces an isomorphism:

$$\tau^{\leq j} S_j \xrightarrow{\simeq} \tau^{\leq j} T.$$